THE SUPPRESSION OF LIGHT SCATTERING IN FERMIONS

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1. INTRODUCTION
The creation of a Fermi degenerate gas on a microchip has opened the door for many experiments on fermionic systems, allowing us to increase our understanding of fermions and ultra-cold atomic systems in general. However, fermionic systems pose many challenges to experimentalists. Unlike a Bose-Einstein condensate, there is no clear indication of a transition to degeneracy in fermions. The existence of a Fermi degenerate gas can only be verified through more subtle signs, such as the existence of Pauli pressure on the atoms. As a result, it is interesting to examine other signatures of Fermi degeneracy. One effect which is unique to degenerate fermions is the suppression of light scattering at low temperatures.

We present a brief overview of the theory of light scattering (Section 2), a survey of previous work on the subject (Section 3), and a discussion of the issues involved in performing numerical calculations (Section 4). We investigate ways to overcome these problems and derive a semiclassical approximation, which yields insight into the physics of scattering in fermions and may prove valuable in future computational work (Section 5). Finally, we investigate a three-level atom which may be more suited to laboratory situations and can enhance the scattering signal (Section 6) and conclude with a discussion of future work that may be undertaken on the subject (Section 7).

2. THEORY OF LIGHT SCATTERING
We now develop a model of light scattering and derive the appropriate analytic expressions for calculating suppression in a Fermi gas. To begin, we consider a collection of $N$ fermions in the ground state $|g\rangle$ of a three-dimensional, anisotropic harmonic potential with Hamiltonian:

$$\hat{H} = \sum_{i=x,y,z} \left( \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m \omega_i^2 \hat{Q}_i^2 \right)$$

An incident photon of wavenumber $k_0$ traveling along the $z$-axis of the trap undergoes Raman scattering off of one of the atoms in the trap. The emitted photon also has wavenumber $k_0$, and it travels in a random direction given by a distribution dependent on whether the light is $\sigma$ or $\pi$-polarized (see Appendix A). For any non-zero scattering angle, this photon imparts a momentum kick to the atom as it decays, and the net momentum change is given by:

$$\Delta k = k_0 \left( -\sin \theta \cos \phi, -\sin \theta \sin \phi, 1 - \cos \theta \right)$$

$$|\Delta k| = 2k_0 \sin(\theta/2)$$

In these expressions, $\theta$ represents the polar angle, while $\phi$ is the azimuth. This change in momentum corresponds to a recoil energy of:
\[ E_R = \frac{\hbar^2 |\Delta k|^2}{2m} = \frac{2\hbar^2 k_0^2 \sin^2(\theta / 2)}{m} \]

We perform a Fermi’s Golden Rule calculation to determine the suppression. Suppose we are considering an initial state \( |m\rangle = |m_x m_y m_z\rangle \) and a final state \( |n\rangle = |n_x n_y n_z\rangle \). In order for scattering to occur in fermions, the initial state must be occupied and the final state must be unoccupied. These probabilities can be determined from the Fermi-Dirac function. We multiply these two terms by the probability of transition from state \( |m\rangle \) to \( |n\rangle \), given that there is a change in momentum given by \( \Delta p = \hbar \Delta k \). Since infinitesimal changes in momentum are generated by the position operator, we define the “boost” operator \( e^{i\Delta p \hat{X} / \hbar} = e^{i\Delta k \hat{X}} \), which shifts the momentum of a given state by \( \hbar \Delta k \). We need to calculate the transition amplitude of this operator between states \( |m\rangle \) and \( |n\rangle \).

Putting these three terms together and normalizing properly, we get:

\[
M_f(\theta, \phi) = \frac{1}{N} \sum_m \sum_n f(m)[1 - f(n)] \langle n | e^{i\Delta k \hat{X}} | m \rangle
\]

\( M \) is the “modification factor”, which is the number of scattering events which occur with blocking statistics divided by the total number of scattering events predicted by classical statistical mechanics. To calculate \( M \), we need to determine an analytic representation of the matrix element in Equation (1). Because the operator \( e^{i\Delta k \cdot \hat{X}} \) gives a shift in momentum, it is easiest to resolve it in the momentum basis. Making the projection into the momentum basis gives us:

\[
\langle n | e^{i\Delta k \cdot \hat{X}} | m \rangle = \int \phi^*_{m}(p + \hbar \Delta k) \phi_m(p) dp
\]

where the \( \phi_m(p) \) are the momentum space representations of the oscillator energy eigenstates. Equation (1) becomes:

\[
M_f(\Omega) = \frac{1}{N} \sum_m \sum_n f(m)[1 - f(n)] \left| \int \phi^*_{n}(p + \hbar \Delta k) \phi_m(p) dp \right|^2
\]

Note that this agrees with the expression given in [1]. The full integration is found in Appendix B and yields the final result in one dimension:

\[
\left| \langle n | e^{i\Delta k \hat{X}} | m \rangle \right|^2 = e^{-a} a^{\Delta n} \left( \frac{p^l}{p + \Delta n} \right)^2 [L^\Delta_n(a)]^2
\]
where  \( p = \min(m, n) \),  \( \Delta n = |m - n| \),  \( a = E_R / \hbar \omega \) and \( L \) is the generalized Laguerre polynomial. The three-dimensional case is simply the direct product of three one-dimensional matrix elements.

This model has some limitations that stem from the initial assumptions made about the system. One very important factor that is currently neglected is the possibility that the excited particle decays back into the “hole” left behind in state \( |m\rangle \) when it was excited. In our analysis, we treat the final state distribution as identical to the initial state distribution, which is true except in the case of decay back into \( |m\rangle \). Due to the complexity of the analysis of this particular case, we neglect it for now and will return to it in future work.

In theory, we now have all of the tools required to evaluate the modification factor numerically; in practice, however, it is very computationally intensive to evaluate 6 infinite sums in any circumstance. Therefore, our preliminary investigations will involve symmetric systems which allow for the reduction in the number of sums required and will also allow us to compare our results with the literature.

3. REVIEW OF PAST WORK
3.1 Spontaneous Emission in an Isotropic Potential

Busch et al. first investigated the theory of light emission in a Fermi gas \([2]\). The system under consideration consists of a collection of \( N \) fermions in a ground state \( |g\rangle \) of an isotropic harmonic potential and one fermion in an excited state \( |e\rangle \). As there is only one fermion in the excited state, quantum statistics are irrelevant and we use the simpler Maxwell-Boltzmann statistics. A photon of wavenumber \( k_0 \) is then released in a random direction. This decay is blocked in the same way as the scattered light in the last section, and the modification factor is calculated in an identical way, with the exception that we use an initial Boltzmann distribution \( P \).

\[
M_f(\Omega) = \frac{1}{N} \sum_m \sum_n P(m)(1 - f(n)) \langle m | e^{i\Delta k \cdot x} | n \rangle^2
\]

Because of the isotropy of the system, we can reduce the six infinite summations to two sums. This is shown in Appendix C. The resulting expression is:

\[
M_f(\Omega) = \sum_N \sum_M (1 - f(N))P(M) \sum_{l=0}^{\min(M,N)} \frac{p!}{(p + \Delta n)!} \left[ L^\Delta n_p(a) \right]^2
\]

where  \( p = \min(M, N) - l \) and  \( \Delta n = N - M \). This can be computed directly to determine the suppression.
Our results are shown below, alongside those from [2] for the purposes of comparison (see Figure 1). The similarity between the two plots gives us confidence that our methods of computation are as valid as those used by Busch.

Figure 1a (left) – Our calculation of the suppression of spontaneous emission in an isotropic trap. In the figure, the value of $r$ is the Fermi energy in units of $\hbar \omega$. $f$ is simply the frequency $\omega/2\pi$.

Figure 1b (right) – The suppression of spontaneous emission in an isotropic trap from [2]. The dashed lines show the results for an analytic, semiclassical approximation that will not be discussed in this paper, but will be examined in future work.

Comparison: Note that the graphs do not agree as $T$ goes to zero. This is due to interpolation beyond the last data point in [2], whereas our results terminate at the final data point.

These results are what one would expect: at colder temperatures, the atoms in the Fermi gas drop into the lowest possible energy levels, and since most of the lower levels are filled, there are fewer decay channels available for the excited atom. This leads to a higher incidence of suppression of spontaneous emission at low temperature. We also see that, as the Fermi energy decreases relative to the recoil energy (which is held fixed at $25\hbar\omega$), there is a significant increase in emission. This is due to the fact that, after emission, most atoms are kicked out of the trap entirely, and so there is almost no suppression in emission.

We cannot directly apply these results to the model derived in Section 2, however, as here we consider only a single atom in the excited state and ignore the effects of the incident photon. Because the excited fermion is the only atom in the excited state, we expect that for low temperatures, the atom will occupy only the lowest energy levels. As a result, we are only looking at the excitation of initial states at low energy, whereas for the case of Raman scattering, we include the possibility that an atom of high energy is excited (in fact, this effect dominates the result). This limits the applicability of Busch’s study, but it remains a useful check of our calculation methods.

3.2 Spontaneous Emission at Low Temperature

We now consider the second part of the investigation in [2], namely spontaneous emission of fermions in a trap with cylindrical symmetry at low temperature. We first exploit the symmetry of the cylindrical trap to reduce the number of directions in the
sum. The derivation is similar to that for an isotropic trap and is found in Appendix D, except we use a Boltzmann distribution for the initial state as in Section 3.1. The result is:

$$M_f(\varOmega) = \sum_{\mathcal{R}, \mathcal{M}, \alpha, \alpha'} \sum_{a} \sum_{n} (1 - f(N_{\alpha} + Z_{n_{\alpha}})) \langle \mathcal{M}_{\alpha} + Z_{n_{\alpha}} | \mathcal{M}_{\alpha} - Z_{n_{\alpha}} \rangle \sum_{n_{\alpha}}^{\infty} \left| \langle \mathcal{M}_{\alpha} | N_{\alpha}, N_{\alpha} - n_{\alpha}, \alpha \rangle \right|^2 \sum_{n_{\alpha}}^{\infty} \left| \langle \mathcal{M}_{\alpha} - Z_{n_{\alpha}} | N_{\alpha}, N_{\alpha} - n_{\alpha}, \alpha \rangle \right|^2$$

(5)

At zero temperature, this expression can be simplified by replacing the Fermi-Dirac function with a step function and placing the excited atom in the ground state (see Appendix E). We get:

$$M_f(\theta) = e^{-a_s (\sin \theta + \chi \cos \theta)} \sum_{n_{\alpha}}^{\infty} \sum_{n_{\alpha}}^{\infty} \frac{a_{s_{n_{\alpha}}}^{n_{\alpha}+n_{\alpha}}}{\chi^{n_{\alpha}+n_{\alpha}} \chi_{n_{\alpha}!n_{\alpha}!}} (\sin \theta)^{2n_{\alpha}} (\cos \theta)^{2n_{\alpha}}$$

We use the value of $\chi = 11$ as the anisotropy parameter $\omega_z / \omega_x$, making the trap stiffer along the $z$-axis. This gives us our graph, seen below, as well as the corresponding graph from [2] (Figure 2):

For low values of the Fermi energy, it appears that there is more suppression along the stiff direction of the trap, whereas this trend is reversed for high Fermi energies. This is due to the fact that the transition matrix element distribution is slightly broader for the stiff direction while both are peaked at $E = E_R$. When a step function is used to replace the Fermi-Dirac function in Equation (1), and the excited state is assumed to be in the lowest possible energy state, the expression for $M$ reduces to a sum over the matrix
elements for $E > E_F$. Since the matrix element distribution is broader in the stiff direction but is normalized identically in both directions, the sum for $E > E_F$ will be lower for $E_F < E_R$ and the result will be a smaller value of $M$ along the stiff direction. The converse occurs for $E_F > E_R$, and we find that the value of $M$ will be greater along the stiff direction.

For non-zero temperature, we cannot use this formula and must return to Equation (5). The results of this calculation are displayed in Figure (3), with the figure from [2] given for the purpose of comparison. They do not agree in the low angle limit, as our figure reaches a maximum above zero for the lowest temperature value, whereas the existing literature places the maximum at zero. We expect that, at very low temperature, the low temperature and zero temperature analyses should yield the same result, meaning that the outermost graph in Figure 2 should be the same as the innermost in Figure 3. This is true for our graph but not for the one from [2]; in fact, the two low-temperature plots in Busch’s paper contradict one another. This indicates an error in their calculation.

3.3 Light Scattering in an Anisotropic Trap
The calculation of suppression in light scattering is only a small change from the case of spontaneous emission: in place of the initial Boltzmann distribution, we use a Fermi-Dirac distribution. In this section, we also include the angular dependence of $\Delta k$ in our calculation.

Calculations for Raman scattering suppression in a Fermi degenerate gas have been made by B. DeMarco and D.S. Jin for the case of a cylindrically symmetric trap with light incident along the axis of symmetry [1]. They consider a trap with axial frequency equal to one-tenth of the radial frequency ($\omega_\perp = 0.1 \omega_r$). We conducted a similar investigation, and the results of both are displayed in Figures 4 and 5. The angular dependence of suppression is logical, as at small angles, there is only a small recoil energy imparted to
the atom, and very few atoms will escape the trap after scattering. This leads to a strong signal of scattering suppression. Also, the amount of light suppressed increases when $T$ decreases, since in this case, the system becomes more fully degenerate, closing up any “holes” in the Fermi sea and leading to more suppression.

\[ f(N,r,t) = \frac{1}{e^{N/r+1} - 1} \]

where $r = \frac{E_F}{\hbar \omega}$ and $t = \frac{k_B T}{E_F} = \frac{T}{T_F}$

where $z$ is the fugacity and $N$ is the energy state of the harmonic oscillator. We note that $f(N,r,t) = f(kN,kr,t)$. Thus, if $N_0$ is the quantum number at which we truncate the sum.
(and $f(N_0, r, t)$ is the occupancy), then $N_0$ grows linearly with the value of $r$ to maintain the same occupancy. This means that, for a given frequency, more terms are required for higher Fermi energies.

If we are trying to reduce the computation time of our program by reducing the number of terms, it is clear that systems with lower Fermi energies are ideal. This means that, in order to rapidly evaluate the sums, we would like to transform our system to one with identical dynamics but lower energies. We attempt to do this by holding the ratio of Fermi energy to recoil energy fixed but varying the value of the Fermi energy with respect to the fundamental frequency $\omega$. This has the effect of scaling the problem down to smaller energy or, identically, increasing $\omega$ while holding the energies fixed. The effect of doing so is seen in Figure 6.

We see that, for frequencies much smaller than the Fermi energy, the suppression remains essentially constant. As a result, the problem is independent of scale in the limit of high Fermi energy, and this is essentially a semi-classical result. In the low-energy limit, however, we find that the suppression plunges rapidly. This is due to the fact that, in this limit, the nature of the quantized energy levels becomes significant to the problem (since $E_F \sim \hbar \omega$).

This example illustrates that, at sufficiently high energies, we may reduce the number of terms in Equation (2) by artificially adjusting the value of the oscillator frequency while holding the other energies constant. This allows us to compute the sums more rapidly to a high degree of accuracy. We must exercise caution, though, to ensure that we never enter the limit of low Fermi energy (compared to $\hbar \omega$), as the approximation will break down in this case and the results will not be correct.

### 4.2 Reducing the Number of Dimensions

More advantageous than the reduction in the number of terms due to scaling is the elimination of some of the sums altogether. Since the complexity of the problem grows exponentially with the number of dimensions, it is clearly beneficial to reduce the dimensionality of the problem as much as possible. Of course, we can only eliminate degrees of freedom when there is an appropriate symmetry to exploit. In both the isotropic and cylindrical cases seen above, the trap symmetry was used to reduce the number of dimensions. In the general case, there are many possible set-ups with no symmetry; for example, we may wish to consider a cylindrical trap with light incident perpendicular to the axis of the trap, or we may consider a trap with different frequencies...
along each axis. There is no symmetry in this problem and a full three-dimensional calculation will result.

It is only feasible to calculate the suppression exactly in a three-dimensional situation for low values of the Fermi energy. However, in initial investigations, we observed that the calculated suppression is independent of both trap geometry and trap alignment. Figure 7(a) consists of two calculations: one performed with an isotropic trap and the second with a cylindrical trap of anisotropy parameter $\chi = 0.1$. Both traps had the same geometric mean frequency. Despite these differences in geometry, the angular dependence of suppression is very similar in both cases. Because differences in geometry seem to have little effect on the suppression of the system, we may approximate a three-dimensional system with an isotropic one with the same mean frequency, resulting in a reduction in the computational intensity required to complete the calculation.

We also find that suppression is independent of trap alignment. Figure 7(b) shows the suppression calculation for two identical cylindrical traps with different alignments. For the first, light is incident along the axis of symmetry, whereas for the second, the incident light is perpendicular to the axis of symmetry. Both, however, yield the same suppression signal. Once again, we may approximate an asymmetric trap alignment with one that is symmetric, allowing us to reduce the complexity of the problem.

4.3 Results for the Lab

Given these two simplifications (reducing the number of terms and the number of degrees of freedom), we are now able to perform the calculations needed in the laboratory. Our experimental set-up consists of an anisotropic harmonic trap with a radial frequency of 826 Hz and an axial frequency of 47 Hz. We assume that there are approximately $2 \times 10^4$ atoms in the trap, and the atoms are excited by an incident light beam of 767 nm. This yields a Fermi energy of $18\hbar\omega$ and a recoil energy of $10.7\hbar\omega$. The angular
dependence of suppression at various temperatures can be seen in Figure 8(a). As expected, we see more suppression at lower angles. This is due to a smaller momentum change (and hence a small recoil energy) for low angles, making it less probable that the atom will escape the Fermi sea. This suppresses the probability of a scattering event. Figure 8(b) illustrates the integrated suppression for two possible atom numbers. As expected, the system with more atoms (higher Fermi energy) exhibits more suppression, because this raises the ratio of Fermi energy to recoil energy, making scattering events less likely. Note also that there is negligible difference (less than 0.1%) between the two above 0.4T/T_F, where thermal effects must obscure the difference in Fermi energy and become the dominant factors in scattering suppression.

5. A SEMICLASSICAL APPROXIMATION
5.1 Derivation of the Semiclassical Approximation
Given the observations of the previous sections on reducing the dimensionality and number of terms in the problem, we conclude that certain quantum aspects of the problem (such as energy level quantization in the harmonic oscillator) are insignificant, and this suggests that a semiclassical approximation might be appropriate for the problem.

On the scale of our problem, the uncertainty in position and momentum is insignificant, and we can assume that a particle can simultaneously hold definite position and momentum. Under this assumption, we can assign a density of states at each point in phase space. This density is given by a continuous form of the Fermi-Dirac function:

$$\rho(q, k) = \frac{1}{(2\pi)^3} \frac{1}{e^{(\epsilon - \mu)/kT} + 1} = \frac{1}{(2\pi)^3} f(\epsilon, \mu)$$

where $f$ is the Fermi-Dirac function, $\epsilon$ is the energy, $\mu$ is the chemical potential, $q$ is the position coordinate and $k$ is the wavevector. The function is normalized to yield $N$
particles when integrated over the whole phase space. Because our system is a harmonic oscillator, we can express the energy as:

\[ \varepsilon = \frac{1}{2} \sum_i \left[ m \omega_i^2 q_i^2 + \frac{\hbar^2 k_i^2}{m} \right] \]  

(7)

We use the same scattering model here as we did in Section 2. Light is scattered via a Raman process, and the atom experiences a net change in momentum of \( \hbar \Delta \mathbf{k} \), including both photon absorption and emission. As before, we perform a Fermi’s Golden Rule calculation: scattering can only occur if there is an atom present in the initial state and there is not an atom present in the final state (otherwise, it would be blocked), and we now use the particle density to determine these values. Because we are considering a continuous set of states, we integrate over all states to determine the modification factor \( M \).

\[ M(\theta, \phi) = \frac{1}{N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho(\mathbf{q}, \mathbf{k}) \left[ 1 - (2\pi)^3 \rho(\mathbf{q}, \mathbf{k} + \Delta \mathbf{k}) \right] d\mathbf{q} d\mathbf{k} \]

\[ M(\theta, \phi) = \frac{1}{N(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\mathbf{q}, \mathbf{k}) \left[ 1 - f(\mathbf{q}, \mathbf{k} + \Delta \mathbf{k}) \right] d\mathbf{q} d\mathbf{k} \]

(8)

Unlike Equation (1), there is no interaction term in for the transition between momentum states in Equation (8). This is because we are assuming that the particles possess definite position and momentum; after a change in momentum of \( \hbar \Delta \mathbf{k} \), the new state of the particle in phase space can be determined exactly using conservation of momentum. This is different from the situation described in Equation (1), as energy states of the oscillator have a distribution of momenta and, as a result, the corresponding matrix element must be included in that calculation.

Ultimately, we get the following expression (the derivation may be found in Appendix F):

\[ M_f(\theta) = \frac{8}{\pi} \int_{-\infty}^{+\infty} dk \int_{0}^{\infty} dx x^{3/2} \frac{1}{1 + z^{-1} \exp\left( x/t + k^2/t \right)} \frac{1}{1 + z \exp\left( -x/t - (k + \Delta k(\theta))^2/t \right)} \]

(9)

Here, \( z \) is the fugacity, \( k \) is the non-dimensionalized wavenumber \( k/k_F \), and \( t \) is the non-dimensionalized temperature \( T = T/F \). Equation (9) is much simpler to evaluate than Equation (2), as it only involves two indefinite integrals (as opposed to six infinite sums). Such an expression may be rapidly evaluated using numerical tools such as Mathematica.

There are a few points worthy of note regarding Equation (9). First, we observe that the trap frequency does not appear in the expression, demonstrating that the problem is independent of energy scale. This means that any system with the same ratio of recoil energy and Fermi energy will exhibit the same behaviour, regardless of the absolute value of these parameters. This agrees precisely with the observation made in Section 4.1,
where for high values of the Fermi energy (relative to the frequency), the exact value of the frequency is irrelevant. Second, we find also that there is no reference to the trap parameters in the equation. This means that, in the region of validity for this approximation, the geometry of the trap is also irrelevant. Again, this agrees with our findings in Section 4.2, where for high values of the Fermi energy, the suppression was independent of trap geometry. Thus, this expression not only gives us a better physical understanding of the process of scattering at high Fermi energies, but it also provides a justification for several of our earlier findings.

5.2 Validity of the Approximation

We now want to test the validity of the approximation under various conditions. Given the results in Section 4.1, we expect that the approximation will fail at low values of the Fermi energy, as we can no longer assume a continuous energy distribution below this point. Figure 9 demonstrates the difference between the semiclassical approximation and a calculation based on Equation (2) for an isotropic trap. As expected, the approximation is valid at high energies, and the graph gives us an idea of the error obtained in using the approximation at lower energies. Of course, it is up to the discretion of the experimenter to determine at what point the error becomes unacceptably high, and so the semiclassical approximation may or may not be useful in various situations.

6. SUPPRESSION IN A THREE-LEVEL SYSTEM

In all of the above calculations, we have considered a simplified, two-level model of the atom. In practice, this is somewhat unrealistic, as there are usually several possible decay channels for an excited atom. Here, we consider an atom which begins in one ground state, is excited by an incident photon, and proceeds to decay into a different hyperfine ground state. We assume initially that there is no decay back into the original ground state. A diagram of the set-up is shown in Figure 10.
This set-up provides an interesting way to enhance the suppression of light scattering. In a two-level system, there will never be a very strong suppression signal, due to the fact that fermions on the edge of the Fermi sea will always be able to scatter light, regardless of the size of the recoil energy. If, however, we prepare a small number of fermions in an initial ground state \( |1\rangle \) and excite these in such a way that they decay into a different, final ground state \( |2\rangle \), then light scattering will be reduced so long as the Fermi energy is larger in the second state. This method ensures that light scattering is blocked for all atoms in the initial state, instead of just those in the middle of the cloud of fermions.

An investigation was conducted for such a three-level system using the parameters for our laboratory given above (radial frequency of 826 Hz, axial frequency of 47 Hz, recoil energy due to 767 nm photons). The results are displayed in Figure 11. As expected, the most pronounced suppression effect occurs when the second state has a high Fermi energy and the first state has a low Fermi energy. In this state, we find that we have a significantly stronger signal than for the two-level atom. We also find that, as the first Fermi energy grows, there is less suppression and the advantage of performing this type of experiment decreases. When the Fermi energies are equal in both states, there is clearly no benefit for exploiting a three-level nature of the atom.

7. FUTURE WORK

There are a number of aspects of the project that could be revisited to determine more accurate results or to delve into more complex aspects of the theory. One major factor that is not considered by this theory is that it does not allow for the excited atom to fall back into its original state (i.e. the “hole” left behind when it was excited). In reality, it is likely that the atom will fall back into this state for small momentum kicks, and this will decrease the suppression of light scattering.

Another possible scenario which has not been explored involves the selective excitation of fermions which are most likely to exhibit the suppression of light scattering. For example, if we only excite those atoms at the centre of a Fermi cloud with a beam of
small recoil energy, nearly all of the excitation/decay paths will be blocked and we will find a strong suppression signal.

There are several opportunities for further study of the physics of scattering suppression in fermions. The use of a Fermi’s Golden Rule calculation is used without justification in the publications on the subject, and it would be instructive to derive Equation (1) from more fundamental principles. This would highlight the assumptions made in Equation (1), as well as guide any further refinements of the theory or of the semiclassical approximation.

8. CONCLUSION
We have used a simple model to study the effects of the Pauli blocking of scattering in cold fermions. We first developed tools to replicate the results of past work and verify the accuracy of our methods, and proceeded to study cases of light scattering with no symmetry. These numerical tools were used to calculate the suppression of light scattering for the experiment underway in our laboratory. Furthermore, by examining the effects of energy scale and geometry on the suppression of light scattering, we were able to develop a semiclassical approximation that is valid in many practical situations and study the accuracy lost in making such an approximation. Finally, we studied the enhancement of suppression in a three-level atomic system, and discussed possible future refinements to our model. All of this has helped us gain a better understanding of the physics of the suppression of scattering, as well as provided information to guide experimental efforts.
APPENDICES

APPENDIX A: DISTRIBUTION OF SCATTERED PHOTONS

We start by assuming that the scattered light will be $\sigma^\pm$ polarized, and will thus have the following distribution:

$$f(\theta, \phi) = 1 + \cos^2 \theta$$  \hspace{1cm} (A1)

We first want to normalize this to convert it into a probability density function.

$$C \int f(\Omega) d\Omega = 1$$

$$C \int \int f(\theta, \phi) \sin \theta d\theta d\phi = 1$$

$$C \int_0^{2\pi} \int_0^\pi (1 + \cos^2 \theta) \sin \theta d\theta d\phi = 1$$

$$2\pi C \int_0^\pi \left(1 + \cos^2 \theta\right) \sin \theta d\theta = 1$$

$$2\pi C \left(\frac{8}{3}\right) = 1$$

$$C = \frac{3}{16\pi}$$

Thus, the probability density function is:

$$f(\Omega) = \frac{3}{16\pi} \left(1 + \cos^2 \theta\right)$$  \hspace{1cm} (A2)

Suppose we want to determine how much the total light scattering is suppressed for a given experimental configuration. We already have an angular distribution of scattering suppression, so the total suppression factor will be given by:

$$M = \int f(\Omega) M_f(\Omega) d\Omega$$  \hspace{1cm} (A3)

This can be evaluated numerically using Mathematica or another program. In the case considered above and under conditions of either spherical or cylindrical symmetry, we can integrate over the azimuthal angle to obtain:

$$M = \frac{3}{8} \int M_f(\theta) (1 + \cos^2 \theta) \sin \theta d\theta$$  \hspace{1cm} (A4)
APPENDIX B: DERIVATION OF MATRIX ELEMENT

We want to derive an analytic expression for the suppression of light scattering for a harmonically trapped Fermi Degenerate Gas. We start with Equation (2) in “Theory of Light Scattering”:

\[ M_f (\Omega) = \frac{1}{N} \sum_m \sum_n [1 - f(E_m, t)] f(E_n, t) \int \phi_n^*(p + \hbar \Delta k) \phi_m(p) dp \]

Here, we use \( f(E, t) \) to represent the Fermi-Dirac function with fugacity \( z \) given by \( f(E, T) = \frac{1}{\exp(E/k_B T) + z + 1} \). Also, \( m = (m_x, m_y, m_z) \) and \( n = (n_x, n_y, n_z) \) give the initial and final quantum states of the harmonic oscillator, respectively, while \( \phi_n(p) \) refers to the momentum-space form of the energy eigenstates. The analytic form of these states in one dimension is given by:

\[ \phi_n(p) = \left( -i \right)^n \frac{e^{-p^2/2\beta^2}}{\sqrt{2^n n! \beta \pi^{1/2}}} H_n(p/\beta), \quad \beta = \sqrt{M \omega \hbar} \]  

(B2)

where \( M \) is the particle mass and \( H_n(x) \) is the Hermite polynomial of degree \( n \) [3].

Because Eq. (B1) can be separated into a product of three integrals of one dimension, we first consider the one-dimensional case and later extend it to three dimensions by taking a direct product. The integrand of Eq. (B1) can be expressed as:

\[ \phi_n^*(p + \hbar \Delta k) \phi_m(p) = \frac{i^n e^{-(p+\hbar \Delta k)^2/2\beta^2}}{\sqrt{2^n n! \beta \pi^{1/2}}} H_n((p + \hbar \Delta k)/\beta) \left( -i \right)^m e^{-p^2/2\beta^2} H_m(p/\beta) \]

(B3)

We can complete the square on the exponent to simplify the expression:

\[ (p + \hbar \Delta k)^2 + p^2 = 2p^2 + 2\hbar \Delta kp + \hbar^2 \Delta k^2 \]
\[ (p + \hbar \Delta k)^2 + p^2 = 2\left(p + \hbar \Delta k/2\right)^2 + \hbar^2 \Delta k^2 / 2 \]

With Eq. (B3), this gives:

\[ \phi_n^*(p + \hbar \Delta k) \phi_m(p) = \frac{(-1)^m i^{m+n} e^{-(p+\hbar \Delta k/2)^2/\beta^2} e^{-\hbar^2 \Delta k^2 / 4\beta^2} H_n((p + \hbar \Delta k)/\beta) H_m(p/\beta)}{\beta \sqrt{2^{m+n} m! n! \pi}} \]

(B4)
We make the substitution $u = (p + h\Delta k / 2) / \beta \rightarrow p = \beta u - h\Delta k / 2$ and $dp = \beta du$ and then perform the required integration over $p$ from Eq (B1).

$$I = \int_{-\infty}^{\infty} \phi_n^*(p + h\Delta k) \phi_n(p) dp$$

$$I = \frac{(-1)^{m+n} i^{m+n} e^{-\frac{h^2\Delta k^2}{4\beta^2}}}{\beta \sqrt{2^{m+n} m! n! \pi}} \int_{-\infty}^{\infty} e^{-u^2} H_n(u + h\Delta k / 2\beta) H_n(u - h\Delta k / 2\beta) \beta du$$

This integral has an analytic form, given in [4]. If we define $p = \min(m,n)$ and $\Delta n = |m - n|$ (and hence $\max(m,n) = p + \Delta n$), $I$ becomes:

$$I = \frac{(-1)^m i^m e^{-\frac{h^2\Delta k^2}{4\beta^2}}}{\sqrt{2^{m+n} m! n! \pi}} \left( 2^{p+\Delta n} \pi^{1/2} p! (\pm h\Delta k / 2\beta)^{\Delta n} L_p^{\Delta_n}\left(\frac{h^2\Delta k^2}{2\beta^2}\right) \right)$$

(B5)

where $L_n^\alpha(x)$ is the associated Laguerre polynomial of degree $n$ and parameter $\alpha$. We are only concerned with the square of the magnitude, so:

$$|I|^2 = \frac{e^{-\frac{h^2\Delta k^2}{2\beta^2}} 2^{2p+2\Delta n} p! p! (\frac{h^2\Delta k^2}{4\beta^2})^{\Delta n} L_p^{\Delta_n}\left(\frac{h^2\Delta k^2}{2\beta^2}\right)^2}{2^{m+n} m! n! \pi}$$

(B6)

We make the simplification that one of $m,n$ will be $p$ and the other will be $p + \Delta n$, meaning that $m + n = 2p + \Delta n$ and $m! n! = p! (p + \Delta n)!$

$$|I|^2 = \frac{e^{-\frac{h^2\Delta k^2}{2\beta^2}} 2^{2p+2\Delta n} p! p! (\frac{h^2\Delta k^2}{4\beta^2})^{\Delta n} L_p^{\Delta_n}\left(\frac{h^2\Delta k^2}{2\beta^2}\right)^2}{2^{2p+\Delta n} p! (p + \Delta n)!}$$

$$|I|^2 = e^{-\frac{h^2\Delta k^2}{2\beta^2}} (\frac{h^2\Delta k^2}{2\beta^2})^{\Delta n} \frac{p!}{(p + \Delta n)!} [L_p^{\Delta_n}\left(\frac{h^2\Delta k^2}{2\beta^2}\right)]^2$$

(B7)

This expression can be simplified if we make the substitution $a = \frac{h^2\Delta k^2}{2\beta^2}$ (B8):

$$|I(m,n,a)|^2 = e^{-a} a^{\Delta n} \frac{p!}{(p + \Delta n)!} [L_p^{\Delta_n}(a)]^2$$

(B9)

What is the interpretation of $a$? Using the definition of $\beta$ in Eq. (B2), we get:

$$a = \frac{h^2\Delta k^2}{2\beta^2} = \frac{h^2\Delta k^2}{2M\omega\hbar} = \frac{E_{\text{recoil}}}{\hbar\omega}$$

(B10)
Thus, the interpretation of the constant $a$ is that it gives the recoil energy imparted to the atom by the incident light in non-dimensional terms (i.e. as a multiple of the energy level difference in the oscillator).

Note that $a$ is a function of $\theta$ due to the angular dependence of $\Delta k$. If the light is incident along the $z$-axis, we have from the geometry of the system:

$$\Delta k = k_0 \left( \langle 0,0,1 \rangle - \langle \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta \rangle \right)$$

where $k_0$ is the wavenumber of the incident and scattered light. Taking the magnitude of this expression gives:

$$\Delta k = 2k_0 \sin(\theta / 2)$$

(B11)

Substituting this into Eq. (B10) yields:

$$a = \frac{4h^2k_0^2}{2M\omega\hbar} \sin^2(\theta / 2) = \frac{4p_R^2}{\hbar\omega} \sin^2(\theta / 2) = \frac{4E_R^2}{\hbar\omega} \sin^2(\theta / 2) = 4a_0 \sin^2(\theta / 2)$$

(B12)

using the fact that the recoil momentum is $p_R = h\Delta k$ and $E = p^2 / 2M$. We also made the substitution $a_0 = E_R / \hbar\omega$, which is a property of the incident light.

Note the difference between $E_{\text{recoil}}$, which is the recoil energy imparted to the atom due to the change in momentum in that direction from both absorption and emission, and $E_R$ which is the recoil energy imparted by one photon.

At this point, we must remember that in eq. (B1), we deal with each spatial dimension separately. This means that it is more correct to state:

$$\left| \langle m_i, n_i, a_i \rangle \right|^2 = e^{-a_i}a_i^{\Delta n_i} \frac{p_i!}{(p_i + \Delta n_i)!} \left[ L_{n_i}^{\Delta n_i} (a_i) \right]^2$$

$$a_i = \frac{\hbar^2 \Delta k_i^2}{2\beta_i^2}$$

where $i = x, y, z$

Note that there is a different recoil energy in each harmonic oscillator direction because the change in momentum is different in each direction. We now take the product of these expressions to determine the true transition matrix elements. The projection of the $\Delta k$ vector onto each direction gives:
\[ \Delta k_x = \Delta k \sin \theta \cos \phi \]
\[ \Delta k_y = \Delta k \sin \theta \sin \phi \]
\[ \Delta k_z = \Delta k \cos \theta \]

When these are substituted into the above expression, we end up with the following relations:

\[ a_{x\text{dir}} = a \sin^2 \theta \cos^2 \phi = 4a_0 \sin^2 \left(\theta / 2\right) \sin^2 \theta \cos^2 \phi \]
\[ a_{y\text{dir}} = a \sin^2 \theta \sin^2 \phi = 4a_0 \sin^2 \left(\theta / 2\right) \sin^2 \theta \sin^2 \phi \]
\[ a_{z\text{dir}} = a \cos^2 \theta = 4a_0 \sin^2 \left(\theta / 2\right) \cos^2 \theta \quad (B13) \]

We must also consider the different trap frequencies in each direction. Suppose that the frequency is given by:

\[ \omega_i = \chi_i \omega \]

For simplicity, we will assign \( \chi_x = 1 \) so that \( \omega \) is the trap frequency in the \( x \) direction and the values of \( \chi_y \) and \( \chi_z \) modify this value for the other directions. For the case of spherical symmetry, we have \( \chi_y = \chi_z = 1 \) and the earlier results should be reproduced at all values of the solid angle \( \Omega \).

How do the values of \( \chi \) affect \( a \)? We return to our definition to obtain:

\[ a = \frac{4E_R}{\hbar \omega} \sin^2 \left(\theta / 2\right) \]
\[ E_R = \frac{ah \omega}{4 \sin^2 \left(\theta / 2\right)} \]

The recoil energy does not change with frequency; this means that a change in \( \omega \) requires an appropriate change in \( a \). Specifically, if \( \omega_{\text{new}} = \chi \omega \) then:

\[ a_{\text{new}} = \frac{4E_R}{\hbar \chi \omega} \sin^2 \left(\theta / 2\right) = \frac{1}{\chi} a \quad (B14) \]

This means that the final version of Eq. (B13) is:
\[ a_x = 4a_0 \sin^2(\theta/2)\sin^2 \theta \cos^2 \phi \]
\[ a_y = \frac{4}{\mathcal{X}_y} a_0 \sin^2(\theta/2)\sin^2 \theta \sin^2 \phi \]  
\[ a_z = \frac{4}{\mathcal{X}_z} a_0 \sin^2(\theta/2)\cos^2 \theta \]

We now turn our attention to the other parts of Eq. (B1), specifically the calculation of the Fermi-Dirac function. We define the following constants:

\[ t = T / T_f \]  
\[ r = k_B T_f / \hbar \omega \]

where \( T_f \) is the Fermi temperature and \( k_B \) is the Boltzmann constant. We use these for notational convenience because

\[ rt = \frac{k_B T}{\hbar \omega} \]

The energy for a harmonic oscillator state (dropping constant terms, because we are only concerned with energy transitions) is:

\[ E = n_x \hbar \omega_x + n_y \hbar \omega_y + n_z \hbar \omega_z \]
\[ E = n_x \hbar \omega + n_y \hbar \mathcal{X}_y \omega + n_z \hbar \mathcal{X}_z \omega \]
\[ E = (n_x + n_y \mathcal{X}_y + n_z \mathcal{X}_z) \hbar \omega \]

If we define \( N = n_x + n_y \mathcal{X}_y + n_z \mathcal{X}_z \) (B18) then we get \( E = Nh \omega \) and hence:

\[ f(N, r, t) = \frac{1}{z^{-1}e^{N/rt} + 1} \]

The fugacity can be calculated using the following implicit formula [5]:

\[ Li_3(z) = -\frac{1}{6t^3} \]

where \( Li_3(z) \) is the polylogarithm function of order three. This is most readily calculated using the root finder of Mathematica. The polylogarithm function is defined over the entire complex plane, but the fugacity should always be purely real for our purposes (Mathematica sometimes outputs values with an imaginary part, but it is negligible and likely due to approximations used in the root finding algorithm).
APPENDIX C: CASE OF SPHERICAL SYMMETRY

Suppose we have a trap that is spherically symmetric, and we include the dipole emission pattern of the incident light. Since the trap is spherically symmetric, we have:

\[ \chi_y = \chi_z = 1 \]
\[ N = n_x + n_y + n_z \]
\[ M = m_x + m_y + m_z \]

The light is scattered at an angle \( \theta \) from the direction of incidence. Note that, because of the symmetry of the trap, the only angular dependence comes from \( \alpha \). This means we can rotate the trap to align the \( z \)-axis with the direction of photon scattering, giving us:

\[ \Delta k_z = \Delta k \]
\[ \Delta k_x = \Delta k_y = 0 \]

From Equation (B15) above, this gives us:

\[ a_x = 0 \]
\[ a_y = 0 \]
\[ a_z = 4a_0 \sin^2(\theta/2) \]  \hspace{1cm} (C1)

We want to see what happens when \( a = 0 \). Equation (B9) reduces to:

\[ |l|^2 = 0^\Delta n \frac{p!}{(p + \Delta n)!} \]  \hspace{1cm} (C2)

since a Laguerre polynomial evaluated at zero is always one. This expression is only non-zero in the case of \( \Delta n = |n - m| = 0 \), in which case it yields one. This means that Equation (C2) further reduces to:

\[ |l|^2 = \delta_{mn} \text{ when } a = 0 \]  \hspace{1cm} (C3)

where \( \delta_{mn} \) is the Kronecker delta.

When we apply this to Equation (2), we find that we get \( m_x = n_x \) and \( m_y = n_y \), according to Equation (C3). If we write the sum out in full, we get (ignoring constants):

\[ M_f(\Omega) = \sum_{n_x} \sum_{n_y} \sum_{n_z} \sum_{m_z} \left[ 1 - f(E_{n_x + n_y + n_z}, t) \right] f(E_{n_x + n_y + n_z}) I(m_z, n_z, a)^2 \]  \hspace{1cm} (C4)
The only time \( n_x \) and \( n_y \) appear are in the expression \( n_x + n_y \), so we make the substitution:

\[
l = n_x + n_y
\]  
(C5)

Also, \( l \) has a certain multiplicity:

<table>
<thead>
<tr>
<th>( n_x )</th>
<th>( n_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( l )</td>
</tr>
<tr>
<td>1</td>
<td>( l - 1 )</td>
</tr>
<tr>
<td>( l - 1 )</td>
<td>1</td>
</tr>
<tr>
<td>( l )</td>
<td>0</td>
</tr>
</tbody>
</table>

Thus, the multiplicity of \( l \) is \( l + 1 \). We can re-write Equation (C4) as:

\[
M_f (\Omega) = \sum_{n_z} \sum_{m_z} \sum_{l} (l+1)(1-f(E_{l+n_z}, l))f(E_{l+m_z})l(m_z, n_z, a)^2
\]  
(C6)

In theory, all three of these are infinite sums. However, as these sums converge, we can ignore all terms above a certain maximum index and still retain the desired degree of accuracy. Because the behaviour at high quantum numbers is dominated by the Fermi-Dirac function, we usually place an upper limit on the values of \( N = l + n_z, M = l + m_z \).

We want to use these values as the indices. Because \( n_x, n_y \geq 0 \), if we select a value of \( N \), we are limited to those choices where \( l \leq N \). The same applies for \( M \), so we get the restriction:

\[
l \leq \min(M,N)
\]

Then, with appropriate substitutions, we get:

\[
M_f (\Omega) = \sum_{n_z} \sum_{m_z} \sum_{l=0}^{\min(M,N)} (l+1)(1-f(E_{N}, l))f(E_{M})l(M-l, N-l, a)^2
\]

\[
M_f (\Omega) = \sum_{n_z} \sum_{m_z} (1-f(E_{N}, l))f(E_{M}) \sum_{l=0}^{\min(M,N)} (l+1)e^{-a\Delta n} \frac{p^l}{(p+\Delta n)^2} \left[L_{p}^{\Delta n}(a)\right]^2
\]  
(C7)

where \( p = \min(N-l, M-l) = \min(M,N)-l \), \( a = 4a_0 \sin^2(\theta/2) \), and \( \Delta n = |(N-l)-(M-l)| = |N-M| \).

We have effectively reduced this to a one-dimensional problem.
APPENDIX D: CASE OF CYLINDRICAL SYMMETRY

We now examine a trap with cylindrical symmetry, where:

\[ \chi_y = 1 \]
\[ N = n_x + n_y + \chi_z n_z \]
\[ M = m_x + m_y + \chi_z m_z \]

As before, light is scattered at an angle \( \theta \), but we cannot rotate the system in the same manner as before. The \( z \)-axis is constrained to lie along the axis of symmetry, but we are free to rotate the trap about this axis. We will rotate the system so that the scattered light is entirely in the \( x \)- and \( z \)-directions – this is equivalent to selecting an azimuthal angle of \( \phi = 0 \). This gives the following values for \( a \):

\[ a_x = 4a_0 \sin^2(\theta/2)\sin^2 \theta \]
\[ a_y = 0 \]
\[ a_z = 4a_0 \sin^2(\theta/2)\cos^2 \theta \]  \hspace{1cm} (D1)

It follows that \( m_y = n_y \) due to the Kronecker delta. We get:

\[ M_f(\Omega) = \sum_{n_x} \sum_{n_y} \sum_{n_z} \sum_{m_x} \sum_{m_y} \sum_{m_z} \left( 1 - f \left( E_{n_x+n_y+\chi_z n_z}, t \right) \right) f \left( E_{n_x+n_y+\chi_z m_z}, t \right) |I(m_x, n_x, a_x)|^2 |I(m_z, n_z, a_z)|^2 \]

\hspace{1cm} (D2)

Let \( N = N_a + N_b \) where \( N_a = n_x + n_y \) and \( N_b = \chi_z n_z \). Similarly, \( M = M_a + M_b \) where \( M_a = m_x + n_y \) and \( M_b = \chi_z m_z \). We use the same substitution as before:

\[ M_f(\Omega) = \sum_{N_a} \sum_{M_a} \sum_{n_z} \sum_{m_z} \left( 1 - f \left( E_{N_a+n_z+n_z}, t \right) \right) f \left( E_{M_a+n_z+m_z}, t \right) |I(m_z, n_z, a_z)|^2 \sum_{n_y=0}^{\min(M_a, M_a)} \left( |I(M_a-n_x, N_a-n_y, a_x)|^2 \right) \]

\hspace{1cm} (D3)

This formula has the same form as Equation (C4), except it involves summation over an extra dimension. This is expected due to the lesser symmetry in this case.

The number of atoms appears in Equation (2), but we need some way to calculate it for the case of cylindrical symmetry. The following formula, which is Equation (5.5) in [5], allows us to perform this calculation:

\[ E_f = \hbar \omega (6 \chi N_{\text{atom}})^{1/3} \]  \hspace{1cm} (D4)
Of course, \( r = E_f / \hbar \omega \), and so we obtain the following:

\[
N_{\text{atom}} = \frac{r^3}{6\chi}
\]  
(D5)

**APPENDIX E: DERIVATION OF ZERO T FORMULA**

In this section, we derive Equation (4) from [2] using our Equation (D3) to confirm the result. We start by listing the assumptions that are used in this derivation:

- The initial state of the particle is thermal and obeys the Maxwell-Boltzmann statistics
- The final state is governed by the Fermi-Dirac statistics, as before
- The system is at absolute zero
- Because we are not considering an atom excited by a photon kick, there is no \( \theta \) – dependence of \( \Delta \vec{k} \).

At zero temperature, the Fermi-Dirac function becomes a step function. This is evident because we may write the function as:

\[
f(E, T) = \frac{1}{e^{(E - E_f)/k_B T} + 1}
\]
(E1)

since, at absolute zero, \( \mu = E_f \) by Sommerfeld’s Lemma [6]. If \( E > E_f \), \( f \to 0 \) and \( f \to 1 \) if \( E < E_f \). Since we are concerned with the value \( 1 - f_n \), we expect all states with \( E < E_f \) to vanish and thus we only sum over those satisfying the condition \( E > E_f \).

Also, \( E = (n_{xy} + \chi n_z) \hbar \omega \) and \( E_f = r \hbar \omega \). This condition becomes:

\[
\begin{align*}
n_{xy} + \chi n_z &> r \\
n_z &> \frac{r - n_{xy}}{\chi}
\end{align*}
\]
(E2)

Since the particle is in a thermal state, it follows that it is necessarily in the ground state. This means that \( m_x = m_y = m_z = 0 \). Because of the symmetry we have established, we have \( m_y = n_y = 0 \) (see above). This means that \( n_{xy} = n_x \), and Equation (E2) becomes:

\[
n_z > \frac{r - n_x}{\chi}
\]
(E3)
We substitute into Equation (D3) to get:

\[ M_f (\Omega) = \sum_{n_x} \sum_{n_z} |I(0, n_z, a_z)|^2 |I(0, n_x, a_x)|^2 \text{ subject to (E3)} \]

\[ M_f (\Omega) = \sum_{n_x=0}^{\infty} \sum_{n_z=0}^{\infty} |I(0, n_z, a_z)|^2 |I(0, n_x, a_x)|^2 \]  \hspace{1cm} (E4)

where \( \tilde{n} = \max\left(0, \frac{r-n_z}{\chi}\right) \)  \hspace{1cm} (E5)

Because there is no angular dependence of \( |\Delta \vec{k}| \), Equations (D1) become:

\[ a_x = a_0 \sin^2 \theta \]
\[ a_y = 0 \]  \hspace{1cm} (E6)
\[ a_z = \frac{1}{\chi} a_0 \cos^2 \theta \]

Combining this with Equation (B9) gives us:

\[ M_f (\Omega) = e^{-a_0 \left( \sin^2 \theta + \chi^{-1} \cos^2 \theta \right)} \sum_{n_z=0}^{\infty} \sum_{n_x=0}^{\tilde{n}} \frac{a_0^{n_x+n_z}}{\chi^{n_z} n_x! n_z!} (\sin \theta)^{2n_z} (\cos \theta)^{2n_x} \]  \hspace{1cm} (E7)

which is equivalent to Equation (4) in [2].

**APPENDIX F: A SEMICLASSICAL APPROXIMATION**

We start with Equations (6-8) in Section 5.1 and re-parameterize the coordinates as follows:

\[ x_i^2 = \frac{1}{2} m \omega_i^2 q_i^2 \]
\[ y_i^2 = \frac{1}{2m} \hbar^2 k_i^2 \]

The differentials of these coordinate transformations are:

\[ dq_i = \frac{1}{\omega_i} \sqrt{\frac{2}{m}} dx_i \]
\[ dk_i = \frac{\sqrt{2m}}{\hbar} dy_i \]
The energy becomes:

\[ \varepsilon = \sum_i \left( x_i^2 + y_i^2 \right) \]

Making this substitution, the integral is:

We want to model the effect of a “momentum boost” in the direction of \( \Delta k \). To simplify the expressions, we may rotate the axes so that this vector points along the coordinate axis \( j \). The result is that the momentum remains invariant in all directions except for \( j \), where we have \( k_j \rightarrow k_j + |\Delta k| \). We may separate this term from the remainder in the exponent of Equation (6) as follows:

\[ \varepsilon - \mu = \sum_i x_i^2 + \sum_{i \neq j} y_i^2 - \left( \mu - y_j^2 \right) = r^2 - \left( \mu - y_j^2 \right) \]

where \( r^2 = \sum_i x_i^2 + \sum_{i \neq j} y_i^2 \)

If we transform all coordinates except for \( y_j \), we get a spherical coordinate system and \( r \) is the radial coordinate. After making this new transformation, we can write:

\[ \rho(q, k) = \rho(r, k_j) \]

Because \( k_j \) describes the momentum along \( \Delta k \), and this is the parameter of most interest to us, we will transform it back from \( y_j \) to \( k_j \). As \( \rho \) is now only dependent on two variables, we can integrate the remaining four out to simplify the expression. To do this, we must find the “surface area” of a spherical shell in 5-space. The formula for the surface area of an n-sphere where \( n \) is odd is:

\[ S_n = \frac{(n-1)/2}{(n-1)!} 2^n \pi^{(n-1)/2} r^{n-1} \]

For \( n = 5 \):

\[ S_5 = \frac{2!}{4!} 5 \pi^2 r^4 = \frac{2^6}{2^3 \cdot 3} \pi^2 r^4 = \frac{8\pi^2}{3} r^4 \]

We get:
\[
\left( \prod_i dx_i \right) \left( \prod_{i \neq j} dy_j \right) dk_j = \left( \frac{8\pi^2}{3} \right) r^4 drdk_j
\]

In terms of the original coordinates \( q_i \) and \( k_i \), we have:

\[
\left( \prod_i dq_i \right) \left( \prod_i dk_i \right) = \left( \frac{2m}{h^2} \right) \left( \frac{2}{m} \right)^{3/2} \left( \frac{1}{\bar{\omega}^3} \right) \left( \prod_i dx_i \right) \left( \prod_{i \neq j} dy_j \right) dk_j = \frac{2^{11/2} \pi^2}{3h^2 \bar{\omega}^3 m^{1/2}} r^4 drdk_j
\]

where \( \bar{\omega} = \left( \prod_i \omega_i \right)^{1/3} \) is the mean frequency.

We can make a further substitution \( w = r^2 \) to get:

\[
\frac{2^{11/2} \pi^2}{3h^2 \bar{\omega}^3 m^{1/2}} r^4 drdk_j = \frac{2^{11/2} \pi^2}{3h^2 \bar{\omega}^3 m^{1/2}} \left( \frac{dw}{2r} \right) dk_j = \frac{2^{9/2} \pi^2}{3h^2 \bar{\omega}^3 m^{1/2}} w^{3/2} dwdk_j
\]

Substituting into Equation (8) gives:

\[
M_f(\theta, \phi) = \frac{2^{3/2}}{3\pi h^2 \bar{\omega} \bar{\omega} m^{1/2}/N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(w, k_j) \left[ g(w, k_j) \right] w^{3/2} dwdk_j
\]

\[
M_f(\theta, \phi) = \frac{2^{3/2}}{3\pi h^2 \bar{\omega} \bar{\omega} m^{1/2}/E_{\tilde{\beta}}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(w, k_j) \left[ g(w, k_j) \right] w^{3/2} dwdk_j
\]

\[
M_f(\theta, \phi) = \frac{2^{5/2} \hbar}{E_{\tilde{\beta}} \bar{\omega} \bar{\omega} m^{1/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(w, k_j) \left[ g(w, k_j) \right] w^{3/2} dwdk_j
\]

In the above expression,

\[
f(w, k) = \left[ 1 + \exp \left( \frac{1}{k_B T} \left( w + \frac{h^2 k_j^2}{2m} - \mu \right) \right) \right]^{-1} = \left[ 1 + z^{-1} \exp \left( \frac{1}{k_B T} \left( w + \frac{h^2 k_j^2}{2m} \right) \right) \right]^{-1}
\]

\[
1 - f(w, k + k_j) = 1 - \left[ 1 + z^{-1} \exp \left( \frac{1}{k_B T} \left( w + \frac{h^2 (k_j + \Delta k)^2}{2m} \right) \right) \right]^{-1}
\]
\[ 1 - f(w, k + k_j) = \left[ 1 + z \exp \left( \frac{-1}{k_B T} \left( w + \frac{\hbar^2 (k_j + \Delta k)^2}{2m} \right) \right) \right]^{-1} \]

We can non-dimensionalize as follows:

\[
\begin{align*}
    u &= \frac{w}{E_F} \\
    dw &= E_F du \\
    K^2 &= \frac{\hbar^2 k_j^2}{2mE_F} \\
    dk_j &= \frac{\sqrt{2mE_F}}{\hbar} dK \\
    \Delta K &= \frac{\hbar^2 \Delta k^2}{2mE_F} \\
    t &= \frac{T}{T_F} = \frac{k_B T}{E_F} \rightarrow \frac{1}{k_B T} = \frac{1}{E_F t}
\end{align*}
\]

This gives:

\[
\begin{align*}
    M_f(\theta, \phi) &= \frac{2^3}{E_F^{3/2} \pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u, K)[1 - f(u, K + \Delta K)]u^{3/2} E_F^{3/2} dudK \\
    M_f(\theta, \phi) &= \frac{8}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u^{3/2} f(u, K)[1 - f(u, K + \Delta K)]u^{3/2} dudK
\end{align*}
\]

where:

\[
\begin{align*}
    f(u, K) &= \left[ 1 + z^{-1} \exp \left( \frac{u}{t} + \frac{K^2}{t} \right) \right]^{-1} \\
    1 - f(w, k + k_j) &= \left[ 1 + z \exp \left( -w \frac{(K + \Delta K)^2}{t} \right) \right]^{-1}
\end{align*}
\]

This is the same expression as Equation (9).
REFERENCES